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Bäcklund transformations and solutions to KdV-type equations

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Abstract. Bäcklund transformations for the non-potential MKdV equation and its Calogero-Degasperis-Fokas modifications are presented and used for finding particular, wave-type solutions to the equations.

1. Introduction

The most powerful method for the solution of nonlinear partial differential equations seems to be the inverse scattering transform. Nevertheless, it has its limitations, namely it yields only regular solutions usually tending to zero for $x \to \pm\infty$. There are, of course, other solutions that do not satisfy the requirements and still may be physically applicable or at least interesting from the mathematical point of view. It is therefore useful to investigate other methods of solutions too. The Bäcklund transformations (BTs) provide a very succesful and relatively simple method.

Recently I found BTs that, to the best of my knowledge, are new and the solutions produced by them are reported in the following sections.

2. Bäcklund transformations for modified MKdV equations

A few years ago Fokas (1980) derived a class of KdV-type equations that should be exactly solvable. The class consists of equations that can be written in one of the following forms

$$u_t + u_{xxx} - K(u_x)^2 = 0, \tag{2.1}$$

$$u_t + u_{xxx} - K(u_x)^3 - F(u)u_x = 0, \tag{2.2}$$

where K is constant and F is a function satisfying

$$F'''(u) = 8KF'(u). \tag{2.3}$$

The equations (2.2) were also derived by a different approach by Calogero and Degasperis (1981).

The equations (2.1) and (2.2) proved to be the only KdV-type equations possessing a certain type of BT (Hlavatý 1985) and explicit forms of the auto-BTs are given in the following.

We shall deal only with equation (2.2), which for obvious reasons may be called the modified MKdV equation. The BT for (2.1) is well known (Wahlquist and Estabrook 1973) and explored.

The auto-BTs for (2.2) are given by (Hlavatý 1985)

$$w_x - bz_x = H(w+bz)(G(w-bz))^{1/2} \tag{2.4}$$

where $w(x, t)$, $z(x, t)$ are supposed to be solutions of (2.2), $b^2 = 1$, and H, G are functions satisfying

$$H'' = \tfrac{1}{2}KH \tag{2.5}$$

$$G''' = 2KG' \tag{2.6}$$

and related to F by the equation

$$F(w) - F(z) = \frac{3}{2}\frac{\mathrm{d}}{\mathrm{d}w}(H^2(w+bz))\frac{\mathrm{d}}{\mathrm{d}w}(G(w-bz)). \tag{2.7}$$

The other equation determining the BT namely $w_t = f(w, z, z_t, z_x, z_{xx})$ can be derived from (2.2) and (2.4). We shall not display its general form which is rather complicated and of no use in what follows, but only its simplified version for $z = z_0 =$ constant that reads

$$w_t = w_x\{F(z_0) + KH^2G - [(H^2)''G - (H^2)'G' + H^2G'']\} \tag{2.8}$$

and will be helpful in the next sections. H, G in (2.8) mean $H(w+bz_0)$, $G(w-bz_0)$ and the prime denotes the differentiation with respect to w.

Given K it is easy to find the functions F, G, H that determine the equation and its BT.

3. MKdV equation

If $K = 0$ and F is a quadratic function then (2.2) can be written in the form of the MKdV equation

$$u_t + u_{xxx} - 6\varepsilon u^2 u_x = 0 \tag{3.1}$$

where ε is a constant. If $\varepsilon > 0$ then the equation has no soliton solutions because the spectrum of the corresponding AKNS system has no discrete part (in contradistinction with the MKdV with $\varepsilon < 0$) and therefore not many explicit solutions of (3.1) with $\varepsilon > 0$ are known (see, e.g., Perelman *et al* 1974).

In correspondence with the general formulae (2.4)-(2.7), the BT for (3.1) is given by

$$w_x - bz_x = (w+bz)[\varepsilon(w-bz)^2 + a]^{1/2}. \tag{3.2}$$

If $bz = z_0 =$ constant then it follows from (2.8) that

$$w_t = w_x(2\varepsilon z_0^2 - a) \tag{3.3}$$

and denoting

$$C := a + 4\varepsilon z_0^2, \qquad \theta(x, t) := (|C|)^{1/2}[x + (2\varepsilon z_0^2 - a)t - x_0] \tag{3.4}$$

we obtain from (3.2) the following simple solutions of (3.1) (for details see appendix).

For $C>0$, $\varepsilon a=\alpha^2>0$

$$w(x, t)=C[2\varepsilon z_0+\alpha \sinh \theta(x, t)]^{-1}-z_0. \tag{3.5}$$

For $a=0$, $\varepsilon>0$

$$w(x, t)=[\varepsilon^{1/2}(x-x)]^{-1}, \tag{3.6a}$$

$$w(x, t)=z_0 \coth[\theta(x, t)/2], \tag{3.6b}$$

$$w(x, t)=z_0 \tanh[\theta(x, t)/2]. \tag{3.6c}$$

For $C>0$, $\varepsilon a=-\alpha^2<0$

$$w(x, t)=C[2\varepsilon z_0+\alpha \cosh \theta(x, t)]^{-1}-z_0. \tag{3.7}$$

For $a=-4\varepsilon z_0^2$

$$w(x, t)=4z_0[1-4\varepsilon z_0^2(x+6\varepsilon z_0^2 t-x_0)^2]^{-1}-z_0. \tag{3.8}$$

For $C<0$, $\varepsilon a=-\alpha^2<0$

$$w(x, t)=C[2\varepsilon z_0+\alpha \sin \theta(x, t)]^{-1}-z_0. \tag{3.9}$$

The superposition formula for the solutions of (3.1)

$$(w_1+w_0)[\varepsilon(w_1-w_0)^2+a_1]^{1/2}+(w_1+w_3)[\varepsilon(w_1-w_3)^2+a_2]^{1/2}$$
$$=(w_2+w_0)[\varepsilon(w_2-w_0)^2+a_2]^{1/2}+(w_2+w_3)[\varepsilon(w_2-w_3)^2+a_1]^{1/2} \tag{3.10}$$

follows from (3.2) and can be used for generation of more complicated solutions.

The solutions (3.5)-(3.6*b*) are obviously singular but (3.6*c*) and (3.7)-(3.9) for $\varepsilon<0$ are regular. The solution (3.7) for $\varepsilon<0$, $z_0=0$ is the well known soliton solution.

Let us remark finally that the BT (3.2) turns out to be more powerful than its counterpart for the potential MKdV because e.g. the variety of solutions generated by (3.2) from a constant solution is larger than that generated by the BT for the corresponding potential MKdV.

4. Modified MKdV equations

In this section we are going to investigate the equation (2.2) with $K\neq 0$, which represents various modifications of the (potential) MKdV equation. We shall omit the simplest case $F(u)=0$ because its BTs are known (Wadati 1973, Lamb 1974) and explored.

Let us start with $K=2k^2>0$. The equation (2.2) then reads

$$u_t+u_{xxx}-2k^2(u_x)^3-\tfrac{3}{2}u_x[f_+\exp(4ku)+f_-\exp(-4ku)+f_0]=0 \tag{4.1}$$

where k, $f_\pm$, f_0 are real constants.

It follows from (2.4)-(2.6) that the BTs for this equation are given by

$$w_x=bz_x+(2k)^{-1}\{h_+\exp[k(w+bz)]+h_-\exp[-k(w+bz)]\}$$
$$\times\{g_+\exp[2k(w-bz)]+g_-\exp[-2k(w-bz)]+g_0\}^{1/2}, \tag{4.2}$$

where $b^2=1$, and from (2.7) the following relations holds:

$$\text{If} \quad b=+1 \quad \text{then} \quad h_\pm^2=\gamma^{-1}f_\pm,\ g_\pm=\gamma. \tag{4.3a}$$

$$\text{If} \quad b=-1 \quad \text{then} \quad h_\pm^2=\gamma^{-1},\ g_\mp=\gamma f_\pm. \tag{4.3b}$$

The constant $g_0\gamma^{-2}$ is a parameter of the BT.

In a similar manner to the previous section, if $z=z_0=\text{constant}$ then

$$w_t=-vw_x, \tag{4.4a}$$

$$v=-\tfrac{1}{2}[f_+\exp(4kz_0)+2h_+h_-g_0+f_-\exp(-4kz_0)]-f_0 \tag{4.4b}$$

and by the substitution

$$W=\exp[2k(w-bz)] \tag{4.5}$$

the equation (4.2) is in the form of the case solved in the appendix. The solutions of (4.1) are then of the form

$$w(x,t)=(2k)^{-1}\log|1/Y(X;A,B,C)-P|+bz_0 \tag{4.6}$$

where Y is given by (A5)-(A13),

$$X=h_+\exp(2kbz_0)[x-vt-x_0], \tag{4.7}$$

$$A=g_+, \qquad B=g_0-2g_+P, \qquad C=g_--g_0P+g_+P^2, \tag{4.8}$$

$$P=h_-\exp(-4kbz_0)/h_+. \tag{4.9}$$

If $K=-2k^2<0$ then (2.2) can be written in the form

$$u_t+u_{xxx}+2k^2(u_x)^3-3\varepsilon u_x\cos(4ku)=0. \tag{4.10}$$

The BT for this equation is given by (cf (4.2))

$$w_x=bz_x+k^{-1}\sin[k(w+bz)](2\varepsilon\cos[2k(w-bz)]+g_0)^{1/2}. \tag{4.11}$$

If $z=z_0=\text{constant}$ then

$$w_t=-vw_x=[\varepsilon\cos(4kz_0)-g_0+f_0]w_x \tag{4.12}$$

and by the substitution

$$W=\tan[k(w-bz)] \tag{4.13}$$

equation (4.11) is again in the form of the case solved in the appendix. The obtained solutions of (4.10) are

$$w(x,t)=k^{-1}\tan^{-1}[1/Y(X;A,B,C)-P]+bz_0 \tag{4.14}$$

where Y is given by (A5)-(A13),

$$X=\cos(2kbz_0)[x-vt-x_0], \tag{4.15}$$

$$A=g_0-2\varepsilon, \qquad B=-2P(g_0-2\varepsilon), \qquad C=g_0(1+P^2)+2\varepsilon(1-P^2), \tag{4.16}$$

$$P=\tan(2kbz_0). \tag{4.17}$$

One can write down a superposition principle similar to (3.10) that comes from the BTs (4.11) and use it for finding more complicated solutions but the procedure is rather complicated and hardly of a practical use.

Regular solutions to (4.1) and (4.10) can be easily found out from (4.6)-(4.9) and (4.14)-(4.17) if the conditions in the end of the appendix are used.

5. Conclusions

We have presented BTs for the equations (2.2) and found some particular solutions to these equations. All of the solutions are wave-type because they were generated by the BTs from constant solutions and most of them are non-soliton solutions that would be difficult to derive by the inverse scattering method.

Generation of more complicated solutions by the superposition principles based on the corresponding BTs is possible, but not very convenient.

Appendix

We are going to solve the equation

$$y' = (p + ry)(a + by + cy^2)^{1/2} \tag{A1}$$

where $y = y(x)$ and p, r, a, b, c are constants, $r \neq 0$, and investigate the regularity of solutions.

Substituting

$$y = 1/Y - p/r \tag{A2}$$

we get

$$Y' = (A + BY + CY^2)^{1/2} \tag{A3}$$

where

$$A = c, \qquad B = b - 2Pc, \qquad C = a - bP + cP^2, \qquad P = p/r. \tag{A4}$$

Solution of this equation depends on the values of A, B, C so that we can solve Y as a function of X, A, B, C:

$$Y(X; A \neq 0, B = 0, C = 0) = A^{1/2}X, \tag{A5}$$

$$Y(X; A, B \neq 0, C = 0) = (BX^2/4 - A)/B, \tag{A6}$$

$$Y(X; A, B, C \neq 0) = [Z(X; C, D) - B]/(2C), \tag{A7}$$

where the function $Z(X; C, D)$ satisfies

$$Z'^2 = C(Z^2 - D), \tag{A8}$$

$$D = B^2 - 4AC, \tag{A9}$$

so that for $C < 0$, $D > 0$

$$Z(X; C, D) = sD^{1/2}\sin[(-C)^{1/2}X], \tag{A10}$$

for $C > 0$, $D > 0$

$$Z(X; C, D) = sD^{1/2}\cosh(C^{1/2}X), \tag{A11}$$

for $C>0,\ D=0$

$$Z(X;C,D)=s\exp(\pm C^{1/2}X), \tag{A12}$$

for $C>0,\ D<0$

$$Z(X;C,D)=s(-D)^{1/2}\sinh(C^{1/2}X), \tag{A13}$$

where $s=\pm 1$.

Inserting (A5)-(A13) into (A2) we obtain various solutions of the equation (A1).

From (A2) it is clear that regular solutions of (A1) are given by $Y(X;A,B,C)$ that are different from zero for all X. They are

(1) (A6) if $A<0$,

(2) (A7), (A10) if $A<0$,

(3) (A7), (A11) if $sB<0$ or $A<0$,

(4) (A7), (A12) if $sB<0$.

For the regularity of (4.6) it is necessary to know which solutions of (A1) are different from zero. It means that $Y(X;A,B,C)\neq 1/P$ for all X that is true for

(1′) (A6) if $A+B/P<0$,

(2′) (A7), (A10) if $A+B/P<0$,

(3′) (A7), (A11) if $a<0$ or $s(B+2C/P)<0$,

(4′) (A7), (A12) if $s(B+2C/P)<0$.

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